

# Topologies induced by group actions

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## Abstract

We introduce some canonical topologies induced by actions of topological groups on groups and rings. For  $H$  being a group [or a ring] and  $G$  a topological group acting on  $H$  as automorphisms, we describe the finest group [ring] topology on  $H$  under which the action of  $G$  on  $H$  is continuous. We also study the introduced topologies in the context of Polish structures. In particular, we prove that there may be no Hausdorff topology on a group  $H$  under which a given action of a Polish group on  $H$  is continuous.

## 0 Introduction

The main motivation for this paper is the following general problem. Suppose  $G$  is a topological group acting on  $X$ , where  $X$  is a set, possibly equipped with some algebraic structure preserved by the action of  $G$ . When does there exist a "nice" topology on  $X$ , such that the action of  $G$  on  $X$  is continuous, and the topology is compatible with the structure on  $X$ ? Clearly, if there is such a topology which is at least  $T_1$ , then, for every element  $x \in X$ , its stabilizer  $G_x$  is closed in  $G$ . On the other hand, if the latter is satisfied, then, by Remark 0.1 below, we can equip  $X$  with a topology under which the action is continuous and which inherits many properties of the given topology on  $G$ .

Now, suppose that  $X$  is equipped with a group structure (preserved by the action of  $G$ ). Then, the topology  $\tau$  defined below usually fails to be a group topology. In Theorem 1.2, we give a description of the finest group topology on  $X$ , under which the action of  $G$  on  $X$  is continuous. Using this description, we give an example of an action of the polish group  $\text{Homeo}([0, 1])$  on a certain group  $H$ , such that there is no Hausdorff group topology on  $H$  under which the action is continuous.

Also, we give in Theorem 1.9 a description of the finest compatible topology in the case of  $X$  being a ring.

By a topological group we will mean a group equipped with a topology, such that the multiplication and the inversion are continuous functions (we do not assume that the topology is Hausdorff).

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**Remark 0.1** Suppose  $G$  is a topological group acting on a set  $X$ . Define  $\mathcal{U} := \{U \cdot x : U \subseteq G, U \text{ is open}, x \in X\}$ . Then, we have:

- (1) The family  $\mathcal{U}$  is a basis of a topology on  $X$ . Denote this topology by  $\tau$ .
- (2) The action of  $G$  on  $(X, \tau)$  is continuous.
- (3) All  $G$ -orbits in  $X$  are clopen in  $\tau$ ; moreover, for every  $x \in X$ ,  $G/G_x \approx G \cdot x$  (where the orbit  $G \cdot x$  is equipped with the topology  $\tau$ ).
- (4)  $(\forall x \in X)(G_x \text{ is closed in } G) \iff \tau \text{ is } T_1 \iff \tau \text{ is Hausdorff}$ .

*Proof.*

(1) Take any  $x_1, x_2 \in X$ , open sets  $U_1, U_2 \subseteq G$  and a point  $y \in U_1x_1 \cap U_2x_2$ . Then,  $y = u_1x_1 = u_2x_2$  for some  $u_1 \in U_1$  and  $u_2 \in U_2$ . For  $i = 1, 2$ , let  $W_i$  be an open neighbourhood of  $e$  in  $G$  such that  $W_i u_i \subseteq U_i$ . Put  $W = W_1 \cap W_2$ . Then,  $Wy = Wu_i x_i \subseteq U_i x_i$  for  $i = 1, 2$ , so we are done.

(2) Take any  $g \in G$ ,  $x, y \in X$  and an open set  $U \subseteq G$  such that  $gx \in Uy$ . Choose  $u \in U$  so that  $gx = uy$ . Let  $U_1$  be an open neighbourhood of  $e$  in  $G$  such that  $U_1 u \subseteq U$ , and let  $V_1, V_2 \subseteq G$  be open neighbourhoods of  $e$  such that  $V_1 g V_2 \subseteq U_1 g$ . Then,  $(V_1 g)(V_2 x) \subseteq U_1 g x = U_1 u y \subseteq Uy$ , which yields the continuity of the action.

(3) By the definition of  $\tau$ , every  $G$ -orbit is open, so also clopen. It is straightforward to check that  $f_x : G/G_x \rightarrow Gx$  given by  $aG_x \mapsto ax$  is a homeomorphism.

(4) First condition implies third by (3) and the fact that the quotient of any topological group (not necessarily Hausdorff) by a closed subgroup is Hausdorff. Third condition implies second trivially, and second implies first by (2).  $\square$

We will denote the topology  $\tau$  from the above remark by  $\tau(X, G)$ .

## 1 Group actions

In this section, we will describe the finest group [ring] topology on  $H$  (where  $H$  is a group or a ring, respectively) under which a given action of a topological group on  $H$  (as automorphisms) is continuous.

First, let us recall a recent result of Bergman, which we use in our construction. Let  $G$  be a group. We will denote the neutral element of  $G$  by  $e$ , and for  $S \subseteq G$  we will write  $S^* = S \cup \{e\} \cup S^{-1}$ . Following the notation from [1], given any  $\mathbb{Q}$ -tuple  $(S_q)_{q \in \mathbb{Q}}$ , put

$$U((S_q)_{q \in \mathbb{Q}}) = \bigcup_{n < \omega} \bigcup_{q_1 < \dots < q_n \in \mathbb{Q}} S_{q_1}^* \dots S_{q_n}^*.$$

Below, we will omit the symbol  $\bigcup_{n < \omega}$  in similar expressions. For a family  $F$  of subsets of  $G$ ,  $F^G$  will denote the collection of all subsets of  $G$  of the form  $\bigcup_{g \in G} g S_g g^{-1}$ , for  $G$ -tuples  $(S_g)_{g \in G}$  of members of  $F$ . We say that a filter on  $G$  converges to  $e$  in a given topology, if every neighbourhood of  $e$  contains a member of the filter. By Lemma 14 and Proposition 15 from [1], we have:

**Fact 1.1** Let  $F$  be a downward directed family of nonempty subsets of  $G$ . Then the sets  $U((S_q)_{q \in \mathbb{Q}})$ , where  $(S_q)_{q \in \mathbb{Q}}$  ranges over all  $\mathbb{Q}$ -tuples of members of  $F^G$ , form a

basis of open neighbourhoods of  $e$  in a group topology  $\mathcal{T}_F$ , which is the finest group topology on  $G$  under which  $F$  converges to  $e$ .

When  $\rho$  is a topology on  $G$ , we will denote by  $\rho^*$  the topology  $\mathcal{T}_F$ , where  $F$  consists of  $\rho$ -open neighbourhoods of  $e$ . In particular, if  $\rho$  is a group topology, then  $\rho^* = \rho$ .

Let  $G$  be a topological group equipped with a topology  $\sigma$ .

**Theorem 1.2** *Suppose  $G$  acts on a group  $H$  as automorphisms. We identify  $G$  and  $H$  with  $\{e\} \times G < H \rtimes G$  and  $H \times \{e\} < H \rtimes G$ , respectively. Put  $T = (D \times \sigma)^*$ , where  $D$  is the discrete topology on  $H$ . Denote by  $T_H$  and  $T_G$  the topologies induced by  $T$  on the subgroups identified with  $H$  and  $G$ , respectively. Then:*

- (1)  $T_G = \sigma$ .
- (2)  $T_H$  is a group topology on  $H$  under which the action of  $G$  on  $H$  is continuous.
- (3) If  $\rho$  is another group topology on  $H$  under which the action of  $G$  on  $H$  is continuous, then  $T_H$  is finer than  $\rho$ .

We will denote the topology  $T_H$  by  $T(H, G)$ .

*Proof.*

(1) We will show that  $T_G = \sigma^*$  (using the description of  $T = (D \times \sigma)^*$  and  $\sigma^*$  given by Fact 1.1), which suffices since  $\sigma = \sigma^*$ . It is easy to see that  $T_G$  is a group topology on  $G$ . Hence, it is enough to show that the neighbourhoods of  $e$  in  $T_G$  are the same as in  $\sigma^*$ .

Take a  $T_G$ -open neighbourhood of  $e$  of the form

$$\begin{aligned} V &= G \cap \bigcup_{q_1 < \dots < q_n \in \mathbb{Q}} \left( \bigcup_{(h,g) \in H \rtimes G} (\{e\} \times U_{(h,g)}^{q_1})^{(h,g)} \right) \dots \left( \bigcup_{(h,g) \in H \rtimes G} (\{e\} \times U_{(h,g)}^{q_n})^{(h,g)} \right) = \\ &= G \cap \bigcup_{q_1 < \dots < q_n \in \mathbb{Q}} \left( \bigcup_{(h,g) \in H \rtimes G} \{(h(u^g h^{-1}), u^g) : u \in U_{(h,g)}^{q_1}\} \right) \dots \\ &\quad \dots \left( \bigcup_{(h,g) \in H \rtimes G} \{(h(u^g h^{-1}), u^g) : u \in U_{(h,g)}^{q_n}\} \right), \end{aligned}$$

where each  $U_{(h,g)}^q$  is a symmetric  $\sigma$ -open neighbourhood of  $e$  in  $G$ . Then,

$$\bigcup_{q_1 < \dots < q_n} \left( \bigcup_{g \in G} g U_{(e,g)}^{q_1} g^{-1} \right) \dots \left( \bigcup_{g \in G} g U_{(e,g)}^{q_n} g^{-1} \right)$$

is a  $\sigma^*$ -open neighbourhood of  $e$  contained in  $V$ .

Conversly, take a  $\sigma^*$ -open neighbourhood of  $e$  of the form

$$W = \bigcup_{q_1 < \dots < q_n} \left( \bigcup_{g \in G} g U_g^{q_1} g^{-1} \right) \dots \left( \bigcup_{g \in G} g U_g^{q_n} g^{-1} \right),$$

where each  $U_g^q$  is a symmetric neighbourhood of  $e$  in  $G$ . For any  $(h, g) \in G$  and  $q \in \mathbb{Q}$ , find a  $\sigma$ -open, symmetric neighbourhood  $U_{(h,g)}^q$  of  $e$ , whose conjugate by  $g$  is contained in  $U_g$  (it can be chosen independently from  $h$ ). Then,

$$G \cap \bigcup_{q_1 < \dots < q_n \in \mathbb{Q}} \left( \bigcup_{(h,g) \in H \rtimes G} (\{e\} \times U_{(h,g)}^{q_1})^{(h,g)} \right) \dots \left( \bigcup_{(h,g) \in H \rtimes G} (\{e\} \times U_{(h,g)}^{q_n})^{(h,g)} \right)$$

is a  $T_G$ -open neighbourhood of  $e$  contained in  $W$ .

(2)  $T_H$  is a group topology since  $H$  is a subgroup of  $H \rtimes G$ , and  $T$  is a group topology on  $H \rtimes G$  by Fact 1.1. For the continuity of the action, take any  $g \in G$ ,  $h \in H$  and a  $T$ -open set  $U$ , such that  $gh \in U \cap H$ . This means that, in  $H \rtimes G$ ,  $(e, g)(h, e)(e, g)^{-1} \in U$ , so we can choose  $T$ -open sets  $U_1$  and  $U_2$ , such that  $(e, g) \in U_1$ ,  $(h, e) \in U_2$  and  $U_1 U_2 U_1^{-1} \subseteq U$ . Then, for any  $g_1 \in U_1 \cap G$  and  $h_1 \in U_2 \cap H$ , we have that  $(e, g_1)(h_1, e)(e, g_1^{-1}) = (g_1 h_1, e)$  belongs to  $U$ , so  $g_1 h_1 \in U \cap H$ . This proves the continuity of the action.

(3) Suppose  $\rho$  is a group topology on  $H$  under which the action of  $G$  on  $H$  is continuous. Then, the product topology  $\rho \times \sigma$  is a group topology on  $H \rtimes G$ , which is coarser than  $D \times \sigma$ , so, by the choice of  $T$ , we have that  $T$  is finer than  $\rho \times \sigma$ . In particular,  $T_H$  is finer than  $\rho$ .  $\square$

Using the above theorem we obtain an explicit formula describing the topology  $T(H, G)$ :

**Corollary 1.3** *With the notation from the above theorem,  $T(H, G)$  has a basis of open neighbourhoods of  $e$  consisting of the sets:*

$$\bigcup_{q_1 < \dots < q_n \in \mathbb{Q}} \{h_1(u_1 h_1^{-1})u_1(h_2(u_2 h_2^{-1}))u_1 u_2(h_3(u_3 h_3^{-1})) \dots u_1 u_2 \dots u_{n-1}(h_n(u_n h_n^{-1})) : \\$$

$$h_i \in H, u_i \in U_{h_i}^{q_i}, u_1 \dots u_n = e\},$$

where  $(U_h^q)_{h \in H, q \in \mathbb{Q}}$  range over all  $H \times \mathbb{Q}$ -tuples of  $\sigma$ -open symmetric neighbourhoods of  $e$  in  $G$ .

*Proof.* By the description of the topology  $T_H$  given in Fact 1.1, we get that it has a basis of open neighbourhoods of  $e$  consisting of the sets:

$$\bigcup_{q_1 < \dots < q_n \in \mathbb{Q}} \{h_1(v_1^{g_1} h_1^{-1})v_1^{g_1}(h_2(v_2^{g_2} h_2^{-1})) \dots (v_1^{g_1} \dots v_{n-1}^{g_{n-1}})(h_n(v_n^{g_n} h_n^{-1})) : h_i \in H,$$

$$g_i \in G, v_i \in U_{(h_i, g_i)}^{q_i}, v_1^{g_1} \dots v_n^{g_n} = e\} =$$

$$= \bigcup_{q_1 < \dots < q_n \in \mathbb{Q}} \{h_1(u_1 h_1^{-1})u_1(h_2(u_2 h_2^{-1})) \dots (u_1 \dots u_{n-1})(h_n(u_n h_n^{-1})) : h_i \in H,$$

$$g_i \in G, u_i \in (U_{(h_i, g_i)}^{q_i})^{g_i}, u_1 \dots u_n = e\},$$

where  $(U_{(h, g)}^q)_{(h, g) \in H \rtimes G, q \in \mathbb{Q}}$  range over all  $(H \rtimes G) \times \mathbb{Q}$ -tuples of  $\sigma$ -open symmetric neighbourhoods of  $e$  in  $G$ . Since the tuples  $(U_{(h, g)}^q)_{(h, g) \in H \rtimes G, q \in \mathbb{Q}}^g$  range over the same set, we can omit the conjugations in the formula. Now, if we replace each  $U_{(h, g)}^q$  by  $U_{(h, e)}^q$  in a tuple  $(U_{(h, g)}^q)_{(h, g) \in H \rtimes G, q \in \mathbb{Q}}$ , then the corresponding neighbourhood of  $e \in H$  will be contained in the original one. Thus, we obtain the same topology when we restrict ourselves to tuples in which  $U_{(h, g)}^q = U_h^q$  does not depend on  $g$ . This gives the conclusion.  $\square$

In Section 2, we will use the description of  $T(H, G)$  that we have obtained to prove the absence of a compatible Hausdorff topology for some classes of Polish group structures (see Proposition 2.8).

Let us keep the notation from above and define  $\lambda(H, G)$  to be the topology on  $H$  in which a set  $U$  is open if for each  $h_1, h_2 \in H$ , the sets  $h_1 U h_2, h_1 U^{-1} h_2$  are open in the topology  $\tau(H, G)$  (defined after Remark 0.1). It is easy to see that if we equip  $H$  with  $\lambda(H, G)$ , then the action of  $G$  on  $H$  is separately continuous, the inversion on  $H$  is continuous and the multiplication on  $H$  is separately continuous. Moreover,  $\lambda(H, G)$  is the finest topology on  $H$  with these properties. Indeed, let  $\xi$  be any other such topology. Take any  $\xi$ -open set  $V$ . Then, for any  $h_1, h_2 \in H$ ,  $h_1 U h_2, h_1 U^{-1} h_2$  are  $\xi$ -open, so also  $\tau$ -open. Hence,  $V$  is  $\lambda(H, G)$ -open.

**Remark 1.4** *In Theorem 1.2, we can replace the discrete topology  $D$  by any topology on  $H$  which is finer than all group topologies under which the action of  $G$  on  $H$  is continuous. Examples of such topologies are  $\tau(H, G)$  and  $\lambda(H, G)$ . However, the simplest description of  $T(H, G)$  we obtain starting from the discrete topology on  $H$ .*

Let us formulate a remark about the topology  $\lambda(H, G)$  defined above.

**Remark 1.5** *If the topology of  $G$  and  $\lambda(H, G)$  are metrizable and Baire, then  $\lambda(H, G) = T(H, G)$*

*Proof.* Let us equip  $H$  with the topology  $\lambda(H, G)$ . Since the multiplication on  $H$  is separately continuous, the inversion on  $H$  is continuous and the action of  $G$  on  $H$  is separately continuous, we get by Theorem 9.14 from [5] that  $H$  is a topological group with the topology  $\lambda(H, G)$ , and the action of  $G$  on  $H$  is continuous. So,  $\lambda(H, G)$  is coarser than  $T(H, G)$ . But  $\lambda(H, G)$  is always finer than  $T(H, G)$  (see the discussion preceeding Remark 1.4), so these topologies are equal.  $\square$

The above remark can be illustrated by the following example.

**Example 1.6** *Let  $G = S_\omega$  be the group of all permutations of  $\omega$ , considered with the product topology (which is Polish, so, in particular, metrizable and Baire). Consider the action of  $G$  on  $H := 2^\omega$ , given by  $g \cdot h = h \circ g^{-1}$ . Then,  $\lambda(H, G)$  is the product topology on  $2^\omega$ , so it coincides with  $T(H, G)$ .*

*Proof.* Clearly  $\lambda(H, G)$  is finer than the product topology. For the converse, let  $U$  be any  $\lambda(H, G)$ -open neighbourhood of  $0 \in H$ . We will show that it contains an open neighbourhood of  $0 \in H$  in the sense of the product topology. Let  $\omega = A \dot{\cup} B \dot{\cup} C$  be a partition of  $\omega$  into three infinite sets. For  $i, j, k \in \{0, 1\}$  define  $h_{i,j,k} \in H$  to be equal to  $i$  on  $A$ , equal to  $j$  on  $B$ , and equal to  $k$  on  $C$ . For  $i, j, k \in \{0, 1\}$ ,  $h_{i,j,k} + U$  contains a  $\tau(H, G)$ -open neighbourhood of  $h_{i,j,k}$  of the form  $[\alpha_{i,j,k}] \cdot h_{i,j,k}$ , where  $\alpha_{i,j,k} : \omega \rightarrow \omega$  is a partial function with a finite domain, and  $[\alpha] = \{\eta \in S_\omega : \alpha \subseteq \eta\}$ . We finish by the following claim:

**Claim 1** Put  $I = \bigcup_{i,j,k \in \{0,1\}} (\text{dom}(\alpha_{i,j,k}) \cup \text{rng}(\alpha_{i,j,k}))$ . Then,  $\{x \in H : x|_I = 0\} \subseteq U$ .

*Proof of Claim 1.* Take any  $x \in H$  such that  $x|_I = 0$ . Notice that we can choose  $i, j, k \in \{0, 1\}$  so that  $(h_{i,j,k} + x)^{-1}[\{0\}]$  and  $(h_{i,j,k} + x)^{-1}[\{1\}]$  are both infinite, and  $i, j, k$  are not all equal. Then, since  $h_{i,j,k}^{-1}[\{0\}]$  and  $h_{i,j,k}^{-1}[\{1\}]$  are also both infinite, and  $h_{i,j,k} + x$  agrees with  $h_{i,j,k}$  on  $I$ , we can find a permutation  $\eta \in [\alpha_{i,j,k}]$ , such that  $\eta \cdot h_{i,j,k} = h_{i,j,k} + x$ . Thus,  $h_{i,j,k} + x \in h_{i,j,k} + U$ , so  $x \in U$ .  $\square$

Now, we aim towards a description of the finest ring topology on  $R$  under which a given action of a topological group on  $R$  is continuous. First, we give a variant of Fact 1.1, in which we are interested in semigroup topologies (i.e. topologies under which the multiplication is continuous) rather than group topologies on  $G$ , but still we assume that  $G$  is a group. For a subset  $S$  of  $G$ , we will write  $S^\# = S \cup \{e\}$ . We define

$$U'((S_q)_{q \in \mathbb{Q}}) = \bigcup_{q_1 < \dots < q_n \in \mathbb{Q}} S_{q_1}^\# \dots S_{q_n}^\#.$$

Then, by a straightforward modification (which is just replacing expressions of the form  $S^*$  by  $S^\#$ ) of the proof of Lemma 14 and Proposition 15 from [1], we obtain:

**Fact 1.7** Let  $F$  be a downward directed family of nonempty subsets of  $G$ . Then, the sets  $U'((S_q)_{q \in \mathbb{Q}})$ , where  $(S_q)_{q \in \mathbb{Q}}$  ranges over all  $\mathbb{Q}$ -tuples of members of  $F^G$ , form a basis of open neighbourhoods of  $e$  in a semigroup topology  $\mathcal{T}'_F$ , which is the finest semigroup topology on  $G$  under which  $F$  converges to  $e$ .

When  $\rho$  is a topology on  $H$ , we will denote by  $\rho^\#$  the topology  $\mathcal{T}'_F$ , where  $F$  consist of  $\rho$ -open neighbourhoods of  $e$ .

Let  $\sigma$  be a fixed group topology on a group  $G$ . Repeating the proof of Theorem 1.2, we obtain:

**Proposition 1.8** Suppose  $G$  acts on a group  $H$  as automorphisms. We identify  $G$  and  $H$  with  $\{e\} \times G < H \rtimes G$  and  $H \times \{e\} < H \rtimes G$ , respectively. Put  $T' = (D \times \sigma)^\#$ , where  $D$  is the discrete topology on  $H$ . Denote by  $T'_H$  and  $T'_G$  the topologies induced by  $T'$  on the subgroups identified with  $H$  and  $G$ , respectively. Then,  $T'_G = \sigma$  and  $T'_H$  is the finest semigroup topology on  $H$  under which the action of  $G$  on  $H$  is continuous. We will denote it by  $T'(H, G)$ .

Now, we are in a position to give a description of the finest topology in the ring case.

**Theorem 1.9** Suppose  $G$  is a group equipped with a group topology  $\sigma$ , acting as automorphisms on a ring  $R$ . Put  $R_1 = R \times \mathbb{Z}$ , and define  $+$  and  $\cdot$  on  $R_1$  by  $(a, k) + (b, l) = (a + b, k + l)$  and  $(a, k) \cdot (b, l) = (ab + l \times a + k \times b, k \cdot l)$ . Clearly,  $G$  acts on  $R_1$  as automorphisms by  $g(a, k) := (g(a), k)$ . Consider the induced action of  $G$  on

$GL_3(R_1)$ . We identify  $R$  with a subset  $\left\{ \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : x \in R \right\}$  of  $GL_3(R_1)$ . Denote by  $T^r(R, G)$  the topology induced on  $R$  by  $T'(GL_3(R_1), G)$ . Then,  $T^r(R, G)$  is the finest ring topology on  $R$  such that the action of  $G$  on  $R$  is continuous.

*Proof.* Let  $T_1$  be the topology induced on  $R_1$  by  $T'(GL_3(R_1), G)$  (we identify  $R_1$  with a subset of  $GL_3(R_1)$  in the same manner as we do with  $R$ ).

**Claim 1**  $T_1$  is the finest ring topology on  $R_1$  under which the action of  $G$  on  $R_1$  is continuous.

First, suppose the claim is proved and let us see that the conclusion of the theorem follows.

By the claim,  $T^r(R, G)$  is a ring topology on  $R$ , and the action of  $G$  on  $R$  equipped with  $T^r(R, G)$  is continuous (as the restriction of the action on  $R_1$ ).

Suppose  $\rho$  is another topology on  $R$  such that the action of  $G$  on  $R$  is continuous. Consider  $R_1$  equipped with the product of  $\rho$  and the discrete topology  $E$  on  $\mathbb{Z}$ . Then, the action  $G$  on  $R_1$  is also continuous and  $R_1$  is a topological ring, so, by the claim,  $T_1$  is finer than  $\rho \times E$ . Hence,  $T^r(R, G)$  (which is equal to the topology induced on  $R$  by  $T_1$ ) is finer than  $\rho$ , so we are done.

*Proof of Claim 1.* First, we will check that  $T_1$  is finer than every ring topology on  $R_1$  under which the action of  $G$  on  $R_1$  is continuous. Let  $\chi$  be any such topology. Let us equip  $GL_3(R_1)$  with the topology  $Z$  induced from the product topology  $\chi^9$  on  $R_1^9$ . Then,  $GL_3(R_1)$  becomes a topological semigroup, and the action of  $G$  on it is continuous. Thus,  $T'(GL_3(R_1), G)$  is finer than  $Z$ , so  $T_1$  is finer than the topology induced by  $Z$  on  $R_1$ , i.e.  $T_1$  is finer than  $\chi$ .

Now, consider  $R_1$  equipped with the topology  $T_1$ . The action of  $G$  on  $R_1$  is continuous (as a restriction of the action on  $GL_3(R_1)$ ) and the addition in  $R$  is continuous (as a restriction of the multiplication in  $GL_3(R_1)$ ). Moreover, the additive inversion in  $R$  is continuous, as it is given by the map

$$\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & -x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is continuous with respect to  $T'(GL_3(R_1), G)$ .

It remains to show that the multiplication on  $R_1$  is continuous. So, we will be done if we show that the map

$$\left( \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \mapsto \begin{pmatrix} 1 & 0 & xy \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is continuous with respect to  $T'(GL_3(R_1), G)$ . The latter follows, since

$$\begin{pmatrix} 1 & 0 & xy \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix},$$

and maps

$$\begin{aligned} & \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \\ & = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \\ & \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \\ & = \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\ & \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & -z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

are continuous.

The proof of the theorem has been completed. □

**Remark 1.10** *In the context of Theorem 1.9, we obtain the same topology on  $R$  if we identify  $R$  with*

$$\left\{ \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : x \in R \right\}.$$

*Proof.* This follows from the fact that

$$\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -x & x \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^2 = \left( \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right)^2$$



is a continuous function (the continuity of the inverse to that map follows from the calculations made at the end of the proof of Theorem 1.9).  $\square$

## 2 Topologies on Polish structures

In this section, we will study the topologies introduced above in the context of Polish structures, which were introduced in [6], and studied also in [2, 3, 4, 7].

**Definition 2.1** *A Polish structure is a pair  $(X, G)$ , where  $G$  is a Polish group acting faithfully on a set  $X$  so that the stabilizers of all singletons are closed subgroups of  $G$ . If  $X$  is equipped with a structure of a group, which is preserved under the action of  $G$ , then we call  $(X, G)$  a Polish group structure. We say that  $(X, G)$  is small if for every  $n < \omega$ , there are only countably many orbits on  $X^n$  under the action of  $G$ .*

The class of small Polish structures contains examples of the form  $(X, \text{Homeo}(X))$  (where  $\text{Homeo}(X)$  is considered with the compact-open topology) for  $X$  being one of the spaces  $[0, 1]^n, S^n$  ( $n$ -dimensional sphere),  $(S^1)^n$  for  $n \in \omega \cup \{\omega\}$ , as well as various other examples, see [6, Chapter 4].

By Remark 0.1, we have:

**Corollary 2.2** (1) *If  $G$  is a Polish group acting on a set  $X$ , then  $(X, G)$  is a Polish structure iff  $\tau(X, G)$  is  $T_1$  iff  $\tau(X, G)$  is completely metrizable.*  
(2) *If  $(X, G)$  is a small Polish structure, then  $\tau(X, G)$  is a Polish topology. In particular,  $(X, G)$  is a Polish  $G$ -space if we equip  $X$  with the topology  $\tau(X, G)$ .*

Let  $(X, G)$  be a Polish structure. For any finite  $C \subseteq X$ , by  $G_C$  we denote the pointwise stabilizer of  $C$  in  $G$ , and for a finite tuple  $a$  of elements of  $X$ , by  $o(a/C)$  we denote the orbit of  $a$  under the action of  $G_C$  (and we call it the orbit of  $a$  over  $C$ ).

A fundamental concept for [6] is the relation of  $nm$ -independence in an arbitrary Polish structure.

**Definition 2.3** *Let  $a$  be a finite tuple and  $A, B$  finite subsets of  $X$ . Let  $\pi_A : G_A \rightarrow o(a/A)$  be defined by  $\pi_A(g) = ga$ . We say that  $a$  is  $nm$ -independent from  $B$  over  $A$  (written  $a \perp_A^{nm} B$ ) if  $\pi_A^{-1}[o(a/AB)]$  is non-meager in  $\pi_A^{-1}[o(a/A)]$ . Otherwise, we say that  $a$  is  $nm$ -dependent on  $B$  over  $A$  (written  $a \not\perp_A^{nm} B$ ).*

By [6, Theorem 2.14], under some assumptions,  $nm$ -dependence in a  $G$ -group  $(H, G)$  can be expressed in terms of the topology on  $H$ :

**Fact 2.4** *Let  $(X, G)$  be a Polish structure such that  $G$  acts continuously on a Hausdorff space  $X$ . Let  $a, A, B \subseteq X$  be finite. Assume that  $o(a/A)$  is non-meager in its relative topology. Then,  $a \perp_A^{nm} B \iff o(a/AB) \subseteq_{nm} o(a/A)$ .*

Using the above fact, we now express the relation of  $nm$ -independence in terms of a family of topologies on  $X$ , without assuming anything about the Polish structure  $(X, G)$ .

**Remark 2.5** *Let  $(X, G)$  be a Polish structure and let  $a, A, B \subseteq X$  be finite. Then  $a \perp_A^{nm} B \iff o(a/B) \subseteq_{nm} o(a/A)$ , where  $X$  is equipped with the topology  $\tau(G_A, X)$  (and the action of  $G_A$  on  $X$  is the restriction of the action of  $G$  on  $X$ ).*

*Proof.* The conclusion follows from Fact 2.4 and Corollary 2.2(2).  $\square$

If  $A$  is a finite subset of  $X$  (where  $(X, G)$  is a Polish structure), we define the algebraic closure of  $A$  (written  $Acl(A)$ ) as the set of all elements of  $X$  with countable orbits over  $A$ . If  $A$  is infinite, we define  $Acl(A) = \bigcup \{Acl(A_0) : A_0 \subseteq A \text{ is finite}\}$ . By Theorems 2.5 and 2.10 from [6],  $nm$ -independence has some nice properties corresponding to those of forking independence in stable first-order theories:

**Fact 2.6** *In any Polish structure  $(X, G)$ ,  $nm$ -independence has the following properties:*

- (0) (Invariance)  $a \perp_A^{nm} B \iff g(a) \perp_{g[A]}^{nm} g[B]$  whenever  $g \in G$  and  $a, A, B \subseteq X$  are finite.
  - (1) (Symmetry)  $a \perp_C^{nm} b \iff b \perp_C^{nm} a$  for every finite  $a, b, C \subseteq X$ .
  - (2) (Transitivity)  $a \perp_B^{nm} C$  and  $a \perp_A^{nm} B$  iff  $a \perp_A^{nm} C$  for every finite  $A \subseteq B \subseteq C \subseteq X$  and  $a \subseteq X$ .
  - (3) For every finite  $A \subseteq X$ ,  $a \in Acl(A)$  iff for all finite  $B \subseteq X$  we have  $a \perp_A^{nm} B$ .
- If additionally  $(X, G)$  is small, then we also have:*
- (4) (Existence of  $nm$ -independent extensions) For all finite  $a \subseteq X$  and  $A \subseteq B \subseteq X$  there is  $b \in o(a/A)$  such that  $b \perp_A^{nm} B$ .

Using Remark 2.5, we can slightly simplify some of the arguments from [6]. For example, we reprove the existence of non-forking extensions in small Polish structures (point 4 of Fact 2.6):

Let  $a \subseteq X$  and  $A \subseteq B \subseteq X$  be all finite. Since  $\tau(X, G_A)$  is Polish, and there are countably many orbits over  $B$ , we can find, by the Baire category theorem, an element  $b \in o(a/A)$ , such that  $o(b/B)$  is non-meager in  $\tau(X, G_A)$ . Then, by Remark 2.5, we get that  $b \perp_A^{nm} B$ .

We will now apply Corollary 1.3 to some of the structures constructed in [3, Chapter 2]. First, we outline the construction of those structures.

Suppose  $(X, G)$  is a Polish structure. Let  $H$  be an arbitrary group. For any  $x \in X$  we consider an isomorphic copy  $H_x = \{h_x : h \in H\}$  of  $H$ . By  $H(X)$  we will denote the group  $\bigoplus_{x \in X} H_x$ . Although  $H(X)$  is not necessarily commutative, we will denote its group action by  $+$ . For any  $y \in H(X)$  there are  $h_1, \dots, h_n \in H \setminus \{e\}$  and pairwise distinct  $x_1, \dots, x_n \in X$  such that  $y = (h_1)_{x_1} + \dots + (h_n)_{x_n}$ . We will then write  $h(y) = \{x_i : h_i = h\}$ .

The group  $G$  acts as automorphisms on  $H(X)$  by

$$g((h_1)_{x_1} + \dots + (h_n)_{x_n}) = (h_1)_{gx_1} + \dots + (h_n)_{gx_n}.$$

It was proved in [3] that with this action  $(H(X), G)$  is a Polish structure, and that if  $H$  is countable, and  $(X, G)$  is small, then also  $(H(X), G)$  is small. Moreover, it was proved there that if, additionally,  $X$  is uncountable, then these structures do not possess any  $nm$ -generic orbits (the notion of an  $nm$ -generic orbit was introduced in [6, Definition 5.3]). On the other hand, [6, Theorem 5.5] states:

**Fact 2.7** *Suppose  $(H, G)$  is a small Polish group structure, where  $H$  is equipped with a topology in which  $H$  is not meager in itself (and the action of  $G$  on  $H$  is continuous). Then, at least one  $nm$ -generic orbit in  $H$  exists, and an orbit is  $nm$ -generic in  $H$  iff it is non-meager.*

From the above theorem and the absence of generics it was concluded that for any non-trivial countable group  $H$  and any small Polish structure  $(X, G)$ , if  $X$  is uncountable then there is no non-meager in itself Hausdorff group topology on  $H(X)$ , such that the action of  $G$  on  $H(X)$  is continuous (in particular, there is no such Polish topology). We strengthen this observation in some cases:

**Proposition 2.8** *Let  $H$  be any non-trivial group and let  $X$  be a compact Hausdorff space containing an open subset homeomorphic to  $(0, 1)^n$  for some non-zero  $n \in \omega \cup \{\omega\}$  (notice that the examples listed after Definition 2.1 satisfy this assumption). Then, there is no Hausdorff group topology on  $H(X)$  under which the action of  $\text{Homeo}(X)$  on  $H(X)$  is continuous (where  $\text{Homeo}(X)$  is considered with the compact-open topology).*

*Proof.* Suppose first that  $X = [0, 1]$ . It is enough to show that the topology  $\rho := T(H([0, 1]), \text{Homeo}([0, 1]))$  is not Hausdorff. Take any  $a \in H \setminus \{e\}$ . We will show that any  $\rho$ -open neighbourhood of  $e \in H([0, 1])$  contains the element  $a_{1/3} - a_{2/3}$ . Let  $W$  be any such neighbourhood and choose (by Corollary 1.3) a  $\rho$ -open set

$$V = \bigcup_{q_1 < \dots < q_n \in \mathbb{Q}} \{h_1(u_1 h_1^{-1})u_1(h_2(u_2 h_2^{-1}))u_1 u_2(h_3(u_3 h_3^{-1})) \dots u_1 u_2 \dots u_{n-1}(h_n(u_n h_n^{-1})) : \\ h_i \in H, u_i \in U_{h_i}^{q_i}, u_1 \dots u_n = e\},$$

such that  $V + V \subseteq W$ . Let  $B_\epsilon(id) \subseteq \text{Homeo}([0, 1])$  be a ball (in the supremum metric) contained in  $U_0^0 \cap U_0^3$ , and choose  $n < \omega$  such that  $1/3n < \epsilon$ . Put

$$h = a_{n/3n} + a_{(n+1)/3n} + \dots + a_{(2n-1)/3n}, h' = -a_{(n+1)/3n} - a_{(n+2)/3n} - \dots - a_{2n/3n}$$

and  $U = U_h^1 \cap U_{h'}^1 \cap U_0^2$ . Notice that  $\{u_0(h - u_1 h), u_0(h' - u_1 h') : u_0 \in B_\epsilon(id), u_1 \in U\} \subseteq V$  (to see this, choose  $q_j = j$  for  $j = 0, 1, 2, 3$ ,  $u_2 = u_1^{-1}$ ,  $u_3 = u_0^{-1}$ ,  $h_0 = h_2 = h_3 = 0$  and  $h_1$  equal to either  $h$  or  $h'$ ). Since  $U$  is open, we can find  $u_1 \in U$  such that  $u_1(k/3n) \in (k/3n, (k+1)/3n)$  for  $k = n, n+1, \dots, 2n-1$ . Then, there is some  $u_0 \in B_\epsilon(id)$  such that  $u_0(k/3n) = k/3n$  and  $u_0 u_1(k/3n) = (2k+1)/6n$  for

$k = n, n+1, \dots, 2n-1$ . So, we get that  $\sum_{k=n}^{2n-1} (a_{k/3n} - a_{(2k+1)/6n}) \in V$ . Similarly, we obtain using  $h'$  that  $\sum_{k=n+1}^{2n} (-a_{k/3n} + a_{(2k-1)/6n}) \in V$ . Thus,  $a_{1/3} - a_{2/3} \in V+V \subseteq W$ .

Now, suppose  $X$  is any space as in the statement. Then, we can find a copy  $F$  of  $[0, 1]^n$  contained in  $(0, 1)^n \subseteq X$ , and an isometric (with respect to a fixed metric on  $F$ ) copy  $I$  of  $[0, 1]$  contained in  $F$ , such that every homeomorphism of  $I$  preserving its endpoints can be extended to a homeomorphism of  $F$  having the same distance from the identity (with respect to the supremum metric) and equal to the identity on the border of  $F$  in  $(0, 1)^n$ . Furthermore, since  $(0, 1)^n$  is open in  $X$ , we can extend such a homeomorphism of  $F$  to a homeomorphism of  $X$  equal to the identity on  $X \setminus F$ . Notice that any open neighbourhood of  $id \in Homeo(X)$  contains  $\{f \in Homeo(X) : f|_{X \setminus int(F)} = id, d(id_F, f|_F) < \epsilon\}$  for some  $\epsilon > 0$ , where  $d$  is the supremum metric. Indeed, by the definition of the compact-open topology, such a neighbourhood is of the form  $\{f \in Homeo(X) : f[K_1] \subseteq W_1, \dots, f[K_l] \subseteq W_l\}$  where  $W_i$ 's are open, and each  $K_i$  is a compact subset of  $W_i$ . Then, it is enough to choose  $\epsilon$  such that for each  $i$  and  $x \in K_i \cap F$ ,  $B_F(x, \epsilon) \subseteq F \cap U_i$ . Now, we can repeat the proof that we gave in the case of  $X = [0, 1]$ . Namely, choosing  $V$  as above (but for an arbitrary  $X$ ), we define  $\epsilon$  to be such that  $\{f \in Homeo(X) : f|_{X \setminus int(F)} = id, d(id_F, f|_F) < \epsilon\} \subseteq U_0^0 \cap U_0^3$  and define  $h, h'$  in the same way as above (identifying  $[0, 1]$  with a subset of  $F$ ). Since  $u_0$  and  $u_1$  can be chosen to preserve endpoints of  $[0, 1]$ , the choice of  $F$  and of the copy of  $[0, 1]$  inside it allows us to repeat the argument.  $\square$

The only known examples of small Polish group structures without  $nm$ -generic orbits are of the form  $H(X)$ . For those of them for which we were able to compute the finest compatible topology, it turned out that it is not Hausdorff. This may suggest that there could be a topological property of a group  $H$  other than being non-meager in itself, which guarantees the existence of  $nm$ -generic orbits in a structure  $(H, G)$ .

**Problem 2.9** *Characterize the existence of  $nm$ -generic orbits in a Polish group structure  $(H, G)$  in terms of topological properties of  $H$ .*

In particular, we can ask:

**Question 2.10** *Does the existence of a Hausdorff group topology on a group  $H$  such that the action of a Polish group  $G$  on  $H$  is continuous imply that the structure  $(H, G)$  has an  $nm$ -generic orbit?*

Also, we do not know whether the converse is true.

**Question 2.11** *Does the existence of  $nm$ -generic orbits in a Polish group structure  $(H, G)$  imply the existence of a compatible Hausdorff topology on  $H$ ?*

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